Math 53: Multivariable Calculus **Sections 102, 103**

Handout for 2020-02-19

Problem 1. For the polar curve $r = \cos^2\theta$, $0 \le \theta \le 2\pi$,

(a) Sketch the curve.
(b) Compute the area enclosed by the curve.
(c) Find the slope of the curve at $\theta = \pi/2$. Hint: you will need to evaluate a limit.

Problem 2. For the parametric curve

$$x = 1 + t^2, \quad y = 2\log_2(t), \quad t \in (1,4)$$

(a) Find a Cartesian equation for the curve.
(b) Sketch the curve.
(c) Find the total area under this curve at above the x-axis.

Problem 3. Let $P = (0,1,1)$ and $Q = (1,0,1)$, and let u, v be their respective position vectors (in other words, the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ where O is the origin). Calculate/describe:

(a) The triple product $u \cdot (v \times u)$.
(b) The area of the parallelogram with sides u, v.
(c) A parametric equation for the line L through P in the direction v.
(d) The distance between Q and L. Hint: this can be done either with calculus or with vector manipulations.

Problem 4. Show that the lines $\mathbf{r}_1(t) = \langle 1+t, 2-2t, 3t\rangle$ and $\mathbf{r}_2(s) = \langle 8+s, 2s+8, 3+s\rangle$ are skew, and then find the distance between the two lines.

Problem 5. The surface $xz - yz = 4$ is a cylinder of some kind. Draw some of its traces, and then decide:

It is a ______________ cylinder comprised of lines with direction vector ______________.

Problem 6. Parametrize the curve of intersection of the surfaces $z = \sqrt{x^2+y^2}$ and $z = 4 - y$. What is the name of this curve?

Problem 7. Consider the curve $\mathbf{r}(t) = \langle 2t, \ln t, t^2\rangle$, $1 \le t \le 4$.

(a) Compute its length.
(b) Find its curvature at $t = 1$.
(c) Find its unit tangent and normal vector, also at $t = 1$. (Using the definition of $\mathbf{N}$ is not so bad here.)
(d) Find the center of the osculating circle to the curve at the point $t = 1$.

Answers

Problem 1. This problem is from Daniel Tataru's FA18 Midterm 1.

(a) Omitted (check with your favorite graphing software)

(b) Use the formula, reducing powers via

$$\cos^4\theta = \frac{1}{4}(1+\cos(2\theta))^2 = \cdots$$

to get the final answer of $\boxed{3\pi/8}$.

Problem 2. This problem is from Kelli Talaska's FA19 Midterm 1.

(a) $x = 1 + 2^y$.

(b) Omitted, but this is useful for the next part.

(c) One way to compute the area is to integrate with respect to y instead of with respect to x, which results in the integral $\int_0^4 (17 - (1+2^y))\,\mathrm{d}y$. Compare to your sketch to try and understand the setup of this integral. The final answer is $\boxed{64 - 15/\ln(2)}$.

Problem 3. This problem is from Daniel Tataru's FA18 Midterm 1.

(a) The triple product is zero because the three vectors are coplanar (indeed, two of them are the same!).

(b) Length of cross product: $\boxed{\sqrt{3}}$.

(c) $\mathbf{r}(t) = \langle 0,1,1\rangle + t\langle 1,0,1\rangle = \langle t, 1, 1+t\rangle$.

(d) Either use vector manipulations or optimize the quantity $d^2 = (t-1)^2 + 1 + t^2$. Final answer $\boxed{\sqrt{3/2}}$.

Problem 4. This problem is from Kelli Talaska's FA19 Midterm 1. The most straightforward way to show that they are skew is to observe that

- The direction vectors $\langle 1,-2,3\rangle$ and $\langle 1,2,1\rangle$ are not multiples of one another, thus the lines are not parallel (also not the same).
- Trying to solve the system $1+t = 8+s, 2-2t = 2s+8, 3t = 3+s$ results in no solutions, so the lines do not intersect.

To find the distance between two skew lines, pick any point A on the first line, any points B on the second line, and then find (the absolute value of) the scalar projection of $\overrightarrow{AB}$ onto a vector orthogonal to both lines. Such a vector can be computed by taking the cross product of the lines' direction vectors: $\langle -8,2,4\rangle$. We may as well take $\langle -4,1,2\rangle$. Then the requisite computation gives $\boxed{16/\sqrt{21}}$.

By the way, as you go through that computation, you actually show once again that the lines are skew. When you take the cross product of the direction vectors and get a nonzero answer, that means the direction vectors are not parallel, and thus the lines are not parallel (also not the same). And when you compute a nonzero final answer, that means the lines are not intersecting (because the distance between intersecting lines is zero!).

Problem 5. This problem is from Kelli Talaska's FA19 Midterm 1. You should find hyperbolic traces and line traces. It is a hyperbolic cylinder comprised of lines with direction vector $\langle 1,1,0\rangle$.

Problem 6. This problem is from Kelli Talaska's FA19 Midterm 1, but on that exam she told the students to use $x = t$ (so without that instruction, the problem is slightly harder). We went over this in section:

$$\mathbf{r}(t) = \langle t, 2 - t^2/8, 2 + t^2/8\rangle.$$

It is a parabola.

For a serious challenge, try doing the problem but with a plane like $z = 1 - y/2$. (The intersection will then be an ellipse, and you will not be able to use any of x, y, z as the parameter t.)

Problem 7. This problem is from Daniel Tataru's FA18 Midterm 1.

(a) Use the formula: $\boxed{15 + \ln 4}$.

(b) Use the formula: $\boxed{2/9}$.

(c) In this particular example, it is not so hard to compute $\mathbf{T}$ as a function of t, and then to differentiate and plug in to find $\mathbf{T}'(1)$. Then $\mathbf{N}$ is just that vector rescaled to have length 1: $\boxed{-1/3, -2/3, 2/3}$.

You could also compute $\mathbf{r}'(1)$ and $\mathbf{r}''(1)$ and then find $\mathbf{r}''(1) - \mathrm{proj}_{\mathbf{r}'(1)}\,\mathbf{r}''(1)$, and rescale that vector to have length 1. I'm not sure which is less work.

(d) The center of the osculating circle at $t = 1$ is

$$\mathbf{r}(1) + \frac{1}{\kappa(1)}\mathbf{N}(1) = \langle 2, 0, 1\rangle + \frac{9}{2}\langle -1/3, -2/3, 2/3\rangle = \langle 1/2, -3, 4\rangle$$

i.e. $\boxed{(1/2, -3, 4)}$.